Exact Results on Potts/Tutte Polynomials for Families of Networks with Edge and Vertex Inflations

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We derive exact relations between the Potts model partition function, or equivalently the Tutte polynomial, for a network (graph) G and a network obtained from G by (i) by replacing each edge (i.e., bond) of G by two or more edges joining the same vertices, and (ii) by inserting one or more degree-2 vertices on edges of G. These processes are called edge and vertex inflation, respectively. The physical effects of these edge and vertex inflations are discussed. We also present exact calculations of these polynomials for families of networks obtained via the operation (ii) on a subset of the bonds of the network. Applications of these results include calculations of some network reliability polynomials. In addition, we evaluate our results to calculate various quantities of structural interest such as numbers of spanning trees, etc., and to determine their asymptotic behavior for large networks.

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I. INTRODUCTION

The Potts model has long been valuable as a system exhibiting many-body cooperative phenomena [1]. On a lattice, or, more generally, on a network (i.e., graph) G, at temperature T, the partition function for this model (in zero external field) is $Z = \sum_{\{\sigma_i\}} e^{-\beta \mathcal{H}}$, where $\beta = 1/(k_B T)$, the Hamiltonian $\mathcal{H} = -J \sum_{e_{ij}} \delta_{\sigma_i \sigma_j}$, J is the spin-spin exchange constant, i and j denote vertices (sites) on G, e_{ij} is the edge (bond) connecting them, and σ_i are classical spins taking values in the set $\{1, ..., q\}$. We use the notation $K \equiv \beta J$ and $v \equiv e^K - 1$. Thus, for the Potts ferromagnet (FM, $J > 0$) and antiferromagnet (AFM, $J < 0$), the physical ranges of v are $v \ge 0$ and $-1 \le v \le 0$, respectively. In general, a graph $G = (V, E)$ is defined by its vertex set, V, and its edge set, E. We denote the number of vertices of G as $n \equiv n(G) \equiv |V|$ and the number of edges of G as $e(G) \equiv |E|$. Z is a polynomial in q and v, as will be evident from Eq. (2.1) below. The Potts model partition function Z is equivalent to a function of considerable interest in modern mathematical graph theory, namely the Tutte polynomial [2]-[4], and we shall therefore often refer to this object in a unified manner as the Potts/Tutte polynomial. The partition function of the zero-temperature Potts antiferromagnet is identical to another function of longstanding interest in graph theory, namely the chromatic polynomial, which counts the number of ways of assigning q colors to the vertices of G subject to the condition that no two adjacent vertices have the same color. These are called proper q-colorings of G. An important property of the Potts antiferromagnet is that for sufficiently large q, it exhibits nonzero entropy per site at zero temperature and is thus an exception to the third law of thermodynamics [5, 6].

An interesting problem involving both statistical mechanics and mathematical graph theory is to relate the Potts/Tutte polynomial calculated for a graph G with the corresponding polynomial calculated for a graph $\tilde{G}$ which is obtained by a specified modification of G. In this paper we will first present a general solution to this problem for two classes of $\tilde{G}$'s, namely those obtained (i) by replacing each edge of G by two or more edges joining the same vertices, and (ii) by inserting one or more degree-2 vertices on each edge of G (where the degree, κ_{v_i}, of a vertex v_i is defined as the number of edges that connect to it). We denote these operations as edge and vertex inflations of G, respectively. In the literature on mathematical graph theory, a vertex inflation of G is also called a homeomorphic inflation of G, and the reverse procedure, of removing one or more degree-2 vertices from edges of G, is called a homeomorphic reduction of G. In the remaining sections of our paper we will present results for Potts/Tutte polynomials of certain families of graphs obtained by vertex inflations for a subset of edges. These families include longitudinal vertex inflations of free, cyclic, and Möbius ladder graphs of arbitrarily great length and a family that we call hammock graphs. These results extend our previous work in two ways: (a) as generalizations to the full Potts/Tutte polynomials of our earlier calculations with S.-H. Tsai of the chromatic polynomials for homeomorphic inflations of these families of graphs in Refs. [7]-[9] and (b) as homeomorphic inflations of our previous calculation of the Potts/Tutte polynomials for ladder strips in Ref. [10].

There are several motivations for this work. For an arbitrary graph G, the calculation of the Potts/Tutte polynomial involves a number of computational steps and a corresponding time that grow exponentially rapidly with $n(G)$ and $e(G)$ (e.g., [11, 12]). Furthermore, there is no known exact closed-form solution for Z for arbitrary q and temperature T on the thermodynamic limit of lattice graphs of dimensionality $d \ge 2$. Hence, it is of fundamental value to carry out exact analytic calculations of Potts/Tutte polynomials on various families of graphs,

such as lattice strip graphs and modifications thereof. Furthermore, special cases of the Potts/Tutte polynomial are of considerable interest in their own right. We have noted the importance of the zero-temperature Potts antiferromagnet (chromatic polynomial). Thus, another motivation for our work, as embodied in point (a) above, is to generalize our previous calculations with S.-H. Tsai [7]-[9] to the broader context of the finite-temperature Potts antiferromagnet and the Potts ferromagnet (the latter also at arbitrary temperature). A central motivation is to investigate the effects of edge and vertex inflations of a graph on the Potts/Tutte polynomial of that graph. This involves the generalization in point (b) above. One particular value of our present work is its demonstration of the usefulness of relating calculations in the context of the Potts model to corresponding calculations for the Tutte polynomial and vice versa. Finally, special cases of our results yield various quantities of interest such as reliability polynomials, numbers of spanning trees, etc. for these families of graphs.

II. SOME GENERAL BACKGROUND

In this section we briefly discuss some necessary background that will be used for our calculations. There is a useful relation that expresses the Potts model partition function $Z(G,q,v)$ as a sum of contributions from spanning subgraphs of G. Here, a spanning subgraph $G'=(V,E')$ has the same vertex set as G and a subset of the edge set E, $E' \subseteq E$. This relation is [13]

$$Z(G,q,v) = \sum_{G' \subseteq G} q^{k_c(G')} v^{e(G')} , \tag{2.1}$$

where $k_c(G')$ denotes the number of connected components in G'. This formula shows that $Z(G,q,v)$ is a polynomial in q and v. For the Potts ferromagnet, Eq. (2.1) enables one to generalize q from the non-negative integers to the non-negative real numbers while keeping $Z(G,q,v)$ positive and hence maintaining a Gibbs measure.

The Tutte polynomial of a graph G, denoted $T(G,x,y)$, is defined by [2]-[4]

$$T(G,x,y) = \sum_{G' \subseteq G} (x-1)^{k_c(G')-k_c(G)} (y-1)^{c(G')} , \tag{2.2}$$

where $c(G')$ is the number of linearly independent cycles (circuits) in G'. Let us define

$$x = 1 + \frac{q}{v} , \quad y = e^K = v+1 , \tag{2.3}$$

so that $q=(x-1)(y-1)$. The relation between $Z(G,q,v)$ and $T(G,x,y)$ follows directly from Eqs. (2.1) and (2.2) and is

$$Z(G,q,v) = (x-1)^{k_c(G)} (y-1)^{n(G)} T(G,x,y) . \tag{2.4}$$

With no loss of generality, we will restrict here to connected graphs G, so $k_c(G)=1$.

For a graph G, let us denote $G-e$ as the graph obtained by deleting the edge e and G/e as the graph obtained by deleting the edge e and identifying the two vertices that were connected by this edge of G. This operation is called a contraction of G on e. From Eq. (2.1), it follows that $Z(G,q,v)$ satisfies the deletion-contraction relation

$$Z(G,q,v) = Z(G-e,q,v) + vZ(G/e,q,v) . \tag{2.5}$$

An analogous deletion-contraction relation holds for $T(G,x,y)$.

III. TYPES OF INFLATIONS OF A GRAPH

Here we discuss in greater detail the edge and vertex inflation of a graph G. As part of our study, we will consider the infinite-length limit of lattice strip graphs G of finite width, and also the thermodynamic limit of lattice graphs G of dimensionality $d \ge 2$. In these cases, we will often use the notation $\{G\}$ to indicate these limits. With no loss of generality, we begin by assuming that G has no multiple edges. We define an edge inflation of G to be a graph obtained by replacing one or more of the edges of G by multiple edges joining the same vertices. Clearly, this leaves the number of vertices $n(G)$ invariant. In particular, it is natural to define a uniform ℓ-fold edge inflation of G as the graph $\mathcal{E}_\ell(G)$ obtained by replacing each edge of G by ℓ edges joining the same two vertices. Thus, the number of edges of $\mathcal{E}_\ell(G)$ is $e(\mathcal{E}_\ell(G)) = \ell e(G)$. A κ-regular graph is a graph with the property that all vertices have the same degree, $\kappa_{v_i} = \kappa \ \forall \ v_i \in V$. Although a general graph is not κ-regular, one can define an average or effective vertex degree κ_{eff} as

$$\kappa_{eff} = \frac{\sum_{v_i \in V} \kappa_{v_i}}{n(G)} . \tag{3.1}$$

Clearly, a uniform ℓ-fold inflation of all of the edges of G multiplies κ_{eff} by ℓ. A remark is in order here concerning loops. A loop is defined as an edge joining a vertex back to itself. This would not occur in the statistical mechanical framework, since it would mean a spin σ_i interacting with itself, and hence we will usually assume that G is a loopless graph.

We define a vertex (i.e., homeomorphic) inflation of G to be a graph obtained by inserting one or more degree-2 vertices on one or several edges of G [14]. This leaves the number of (linearly independent) circuits in G, $c(G)$, invariant. Homeomorphic inflations and reductions of graphs have been of interest in the study of chromatic and Tutte polynomials [7]-[9], [15]-[21] and have also been studied in the context of "decorated" spin models [22]. As with edge inflations, it is natural to define a uniform ℓ-fold vertex inflation $\mathcal{V}_\ell(G)$ as the graph obtained by inserting ℓ degree-2 vertices on each edge of G. For the special case where G is a section of a regular lattice, it is also natural to consider edge or homeomorphic inflation of edges forming a subset of lattice vectors. For example,

for a d-dimensional Euclidean lattice $\mathbb{E}^d$, one can consider the case in which the edge or vertex inflation is performed on the edges along one or more lattice vectors $\hat{e}_j$, where j takes values in a subset of $\{1, ..., d\}$.

IV. RELATIONS FOR UNIFORM EDGE AND VERTEX INFLATIONS

A. Edge Inflation

The effect of a uniform edge inflation on the Potts model partition function can be determined from an analysis of the Potts Hamiltonian. As we edge-expand G to $\mathcal{E}_\ell(G)$, i.e., replace each edge by ℓ edges joining the same vertices, we induce the change in the Hamiltonian

$$\mathcal{H} = -J \sum_{e_{ij}} \delta_{\sigma_i \sigma_j} \to -\ell J \sum_{e_{ij}} \delta_{\sigma_i \sigma_j} \ , \tag{4.1}$$

i.e., $J \to \ell J$, and hence

$$y \to y_{e,\ell} = y^\ell \ , \tag{4.2}$$

where the subscripts e, ℓ refer to the ℓ-fold edge inflation. Equivalently,

$$v \to v_{e,\ell} = (v+1)^\ell - 1 \ , \tag{4.3}$$

where $v_{e,\ell} \equiv y_{e,\ell} - 1$. This proves the following relation connecting Z on G with Z on $\mathcal{E}_\ell(G)$:

$$Z(\mathcal{E}_\ell(G), q, v) = Z(G, q, v_{e,\ell}) \ . \tag{4.4}$$

We next determine the effect of this edge inflation on the Tutte polynomial. Given the transformation (4.2) and the fact that q does not change, so that

$$q = (x-1)(y-1) = (x_{e,\ell} - 1)(y_{e,\ell} - 1) \ , \tag{4.5}$$

we have

$$x_{e,\ell} = 1 + \frac{(x-1)(y-1)}{y^\ell - 1} = 1 + \left(\frac{x-1}{\sum_{j=0}^{\ell-1} y^j} \right) \ . \tag{4.6}$$

Combining these transformations of variables with Eq. (2.4), we derive the relation connecting the Tutte polynomials of G and of $\mathcal{E}_\ell(G)$, namely

$$T(\mathcal{E}_\ell(G), x, y) = T(G, x_{e,\ell}, y_{e,\ell}) \ , \tag{4.7}$$

where $x_{e,\ell}$ and $y_{e,\ell}$ were given in Eqs. (4.6) and (4.2). Note that the effect of the edge inflation on the Potts partition function is simpler than the effect on the Tutte polynomial, since in the former case, only one of its variables is modified, namely, $v \to v_{e,\ell}$, whereas in the latter case, both of its variables are modified, as $x \to x_{e,\ell}$ and $y \to y_{e,\ell}$.

Although our main focus is on the Potts model with spin-spin exchange constants J that are independent of the edges e_{ij}, we note parenthetically that one can consider a generalization in which the J_{ij} are different for each edge of G, e_{ij}. From the original 1944 Onsager solution of the two-dimensional Ising model [23], a large number of studies of spin models have dealt with the general case with different spin-spin exchange constants for different lattice directions. Studies of spin models with spin-spin exchange constants J_{ij} that can be different (in magnitude and sign) for each edge, e_{ij} were motivated by early work on spin glasses [24]. In this case, where J_{ij}'s depend on e_{ij}, the transformation of the Hamiltonian becomes

$$\mathcal{H} = -\sum_{e_{ij}} J_{ij} \delta_{\sigma_i \sigma_j} \to -\ell \sum_{e_{ij}} J_{ij} \delta_{\sigma_i \sigma_j} \ . \tag{4.8}$$

Hence, defining $K_{ij} \equiv \beta J_{ij}$ and $v_{ij} \equiv e^{K_{ij}} - 1$, we have

$$v_{ij} \to v_{ij,e,\ell} = (v_{ij} + 1)^\ell - 1 \ . \tag{4.9}$$

Denoting $\{v\}$ as the set of v_{ij}'s, we then have

$$Z(\mathcal{E}_\ell(G), q, \{v\}) = Z(G, q, \{v_{e,\ell}\}) \ . \tag{4.10}$$

Since $J < 0$ for the Potts antiferromagnet, it follows that, as $T \to 0$, $K \to -\infty$ (and $v \to -1$). Hence, in this limit, the only contributions to the partition function are from spin configurations in which adjacent spins have different values. The resultant $T = 0$ Potts AFM partition function is therefore precisely the chromatic polynomial $P(G, q)$ of the graph G counting the number of proper q-colorings of G:

$$Z(G, q, -1) = P(G, q) \ . \tag{4.11}$$

This is equivalent, via Eq. (2.4), to the relation

$$P(G, q) = (-q)^{k_c(G)} (-1)^{n(G)} T(G, 1-q, 0) \ . \tag{4.12}$$

The minimum number of colors necessary for a proper q-coloring of G is the chromatic number $\chi(G)$. For $q > \chi(G)$, $P(G, q)$ grows exponentially with n, leading to a ground state degeneracy per vertex, $W(\{G\}, q) > 1$, where $W(\{G\}, q) = \lim_{n\to\infty} P(G, q)^{1/n}$. The ground state entropy per vertex of the Potts model on $\{G\}$ is $S_0(\{G\}, q) = k_B \ln[W(\{G\}, q)]$. In Refs. [6], [7]-[9], [16], we applied our exact results on chromatic polynomials to study the phenomenon of nonzero ground state entropy per site in Potts antiferromagnets. These exact results complement other approaches to studying S_0, such as rigorous bounds, series, and Monte Carlo measurements [25, 26].

It is clear from the definition of the chromatic polynomial that $P(G, q)$ does not change if one replaces any edge of G by two or more edges joining the same vertices. In particular, for the case of an ℓ-fold uniform edge inflation of G,

$$P(\mathcal{E}_\ell(G), q) = P(G, q) \ . \tag{4.13}$$

This is also clear analytically from Eqs. (4.3) and (4.4), since the condition that $v = -1$ implies that $v_{e,\ell} = -1$.

B. Vertex Inflation

To analyze the effect of a vertex inflation of a graph G, it is again convenient to start with the Potts model formulation. For simplicity, we assume here that G does not have any multiple edges; it is straightforward to extend our calculation to the case of multiple edges. We use the fact that in the basic expression $Z = \sum_{\{\sigma_i\}} e^{-\beta\mathcal{H}}$, one can perform the summations over the spins that are located at these degree-2 vertices. Let us consider an edge, e_{ij}, and insert a degree-2 vertex v_a (and its associated spin, σ_a) on this edge. Then

$$\begin{aligned} & \sum_{\sigma_i,\sigma_a,\sigma_j} (1+v\delta_{\sigma_i\sigma_a})(1+v\delta_{\sigma_a\sigma_j}) \\ & = \sum_{\sigma_i,\sigma_a,\sigma_j} \Big[1+v(\delta_{\sigma_i\sigma_a}+\delta_{\sigma_a\sigma_j})+v^2\delta_{\sigma_i\sigma_j}\Big] \ . \end{aligned} \tag{4.14}$$

Now, carrying out the summation over σ_a, we find that if $\sigma_i = \sigma_j$, then the result is $q+2v+v^2$, while if $\sigma_i \neq \sigma_j$, then the result is $q+2v$. Therefore,

$$\begin{aligned} & \sum_{\sigma_i,\sigma_a,\sigma_j} (1+v\delta_{\sigma_i\sigma_a})(1+v\delta_{\sigma_a\sigma_j}) \\ & = (q+2v)\sum_{\sigma_i,\sigma_j}(1+v_{v,1}\delta_{\sigma_i\sigma_j}) \ , \end{aligned} \tag{4.15}$$

where

$$v_{v,1} = \frac{v^2}{q+2v} \tag{4.16}$$

and the subscript $v,1$ refers to the insertion of one additional vertex on the edges. Performing this summation for each edge, we derive the relation

$$Z(\mathcal{V}_1(G),q,v) = (q+2v)^{e(G)} Z(G,q,v_{v,1}) \ . \tag{4.17}$$

This operation can be performed iteratively ℓ times, thereby giving a relation between Z on $\mathcal{V}_\ell(G)$ and Z on G. For example, for $\ell = 2$, we have

$$\begin{aligned} v_{v,2} &= \frac{v_{v,1}^2}{q+2v_{v,1}} \\ &= \frac{v^4}{(q+2v)(q^2+2qv+2v^2)} \ , \end{aligned} \tag{4.18}$$

and so forth for higher values of ℓ. We can thus relate Z on $\mathcal{V}_\ell(G)$ to Z on G.

A basic problem in graph theory is the enumeration of discretized flows on the edges of a (connected) G that satisfy flow conservation at each vertex, i.e. for which there are no sources or sinks. One arbitrarily chooses a direction for each edge of G and assigns a discretized flow value to it. The value zero is excluded, since it is equivalent to the edge being absent from G; henceforth, we take a q-flow to mean implicitly a nowhere-zero q-flow. The flow on each edge can thus take on any of $q-1$ values modulo q. The flow or current conservation condition is that the flows into any vertex must be equal, mod q, to the flows outward from this vertex. These are called q-flows on G, and the number of these is given by the flow polynomial, $F(G,q)$. This is a special case of the Tutte polynomial for $x=0$ and $y=1-q$ or equivalently, in terms of Potts model variables, $v=-q$:

$$F(G,q) = (-1)^{c(G)}\, T(G,0,1-q) \ . \tag{4.19}$$

As is clear from the definition of the flow polynomial, adding or removing a degree-2 vertex from an edge of G does not change the number of allowed q-flows on G, so

$$F(\mathcal{V}_\ell(G),q) = F(G,q) \ . \tag{4.20}$$

As is evident from Eq. (4.16), the condition $v=-q$ implies that $v_{v,1} = -q$ and hence, more generally, that $v_{v,\ell} = -q$ for arbitrary $\ell \ge 1$. Recall that the number of cycles in G is unchanged by this homeomorphic inflation: $c(\mathcal{V}_\ell(G)) = c(G)$.

V. PHYSICAL EFFECTS OF EDGE AND VERTEX INFLATION

A. General

An important question concerns the effect of edge and vertex inflations on the physical properties of the Potts model. First, one may investigate these effects for the q-state Potts model on (the thermodynamic limit of) a regular lattice graph of dimension $d \ge 2$, where the ferromagnetic version of the model has a finite-temperature order-disorder phase transition, and, depending on the lattice type and the value of q, the Potts antiferromagnet may have a finite-temperature order-disorder phase transition. In particular, one can study the effect on the phase transition (pt) temperature T_{pt} for lattices where the equation for this quantity is known exactly. However, aside from the $q=2$ Ising case on two-dimensional lattices, there is no known exact closed-form solution for the free energy of the q-state Potts model at arbitrary temperature on these lattices with $d \ge 2$. There is thus also some interest in studying the effect of edge and vertex inflation for infinite-length limits of quasi-one-dimensional lattice strips, where one can obtain exact closed-form expressions for the free energy for arbitrary q and T. Of course, a spin model with short-ranged interactions, such as the Potts model, does not have any finite-temperature phase transition on an infinite-length quasi-one-dimensional lattice strip. Nevertheless, one interesting application of exact solutions for the free energy and thermodynamic quantities on these strips is that one can study their dependence (and the dependence of the associated Tutte polynomials) on graphical properties, in particular, on the edge or vertex inflation.

In accordance with our notation above for the dimensionless inverse temperature $K \equiv J/(k_BT)$, we define

$K_{pt} \equiv J/(k_B T_{pt})$. We also introduce some notation that we will use below. We define the shifted values of T_{pt} due to a uniform ℓ-fold edge inflation (symbolized by the subscript e, ℓ) and a uniform ℓ-fold vertex inflation (symbolized by the subscript v, ℓ) by $T_{pt,e,\ell}$ and $T_{pt,v,\ell}$. We thus denote $K_{pt,e,\ell} \equiv J/(k_B T_{pt,e,\ell})$ and $K_{pt,v,\ell} \equiv J/(k_B T_{pt,v,\ell})$.

B. Effect on T_{pt} Due to Edge Inflation

A general result is that for the Potts model on (an infinite) regular lattice graph $\{G\}$ with dimensionality $d \ge 2$ that has a finite-temperature phase transition (of either ferromagnetic or antiferromagnetic type, and either first or second order) at a temperature T_{pt}, a uniform ℓ-fold inflation of all edges of $\{G\}$ has the effect of multiplying T_{pt} by the factor ℓ, i.e.,

$$T_{pt,e,\ell} = \ell T_{pt} \ , \quad K_{pt,e,\ell} = \frac{K_{pt}}{\ell} \ . \tag{5.1}$$

Analytically, this follows because the uniform ℓ-fold edge inflation of $\{G\}$ changes J to ℓJ and T_{pt} is proportional to J. Physically, it follows because replacing J by ℓJ with $\ell \ge 2$ strengthens the spin-spin interaction and hence makes possible the onset of long-range magnetic order in the presence of greater thermal fluctuations, i.e., at a higher temperature.

As an example, consider the Potts ferromagnet on the (thermodynamic limit of the) square lattice, $\{sq\} = \mathbb{E}^2$. Denoting the spin-spin exchange constants in the two lattice directions $\hat{e}_i$, $i = 1, 2$, as J_i, with $K_i = \beta J_i$ and $v_i = e^{K_i} - 1$, we recall the well-known equation for the phase transition temperature, namely [1],

$$v_1 v_2 = q \ . \tag{5.2}$$

Let us initially assume $K_1 = K_2 \equiv K$ and thus $v_1 = v_2 \equiv v$, so Eq. (5.2) becomes $v^2 = q$. The dimensionless inverse phase transition temperature K_{pt} is then given by

$$K_{pt} = \ln(1 + \sqrt{q}\ ) \ . \tag{5.3}$$

Now let us carry out an ℓ-fold edge inflation on the lattice. Denoting the resultant inverse phase transition temperature in an obvious notation as $K_{pt,e,\ell}$, we have

$$K_{pt,e,\ell} = \frac{1}{\ell} \ln(1 + \sqrt{q}\ ) = \frac{K_{pt}}{\ell} \ . \tag{5.4}$$

One can also consider the effect of an edge inflation on all edges along a subset of lattice directions. The effect is simplest for the ferromagnetic case, since this does not involves competing interactions or frustration. This edge inflation along a subset of lattice directions strengthens the net spin-spin interaction and therefore makes possible the ordering associated with the phase transition in the presence of greater thermal fluctuations. Let us denote $T_{pt,es,\ell}$ as the shifted phase transition temperature after an ℓ-fold edge inflation along a subset s of the lattice directions, and similarly denote $K_{pt,es,\ell} \equiv J/(k_B T_{pt,es,\ell})$. Then the reasoning above yields the inequality $T_{pt,es,\ell} > T_{pt}$ for $\ell \ge 2$. As an example, we again consider the (infinite) square lattice $\{sq\}$ and perform an edge inflation with $\ell = 2$ for edges in one of the two lattice directions, say $\hat{e}_2$, so that $K_1 = K$, $K_2 = 2K$. The equation for the shifted inverse phase transition temperature is given by $v^2(v + 2) = q$, with physical solution

$$K_{pt,e2,2} = \ln\left[\frac{1}{3}\Big\{A^{1/3} + 4A^{-1/3} + 1\Big\}\right] \ , \tag{5.5}$$

where the subscript $e2$ means inflation along edges along $\hat{e}_2$ and

$$A = \frac{1}{2}\Big[27q - 16 + 3\sqrt{3q(27q - 32)}\ \Big] \ . \tag{5.6}$$

The resultant $K_{pt,e2,2} < K_{pt}$, i.e., $T_{pt,e2,2} > T_{pt}$, in agreement with the general argument given above. For example, for $q = 2$, $K_{pt} = \ln(1+\sqrt{2}\ ) \simeq 0.88137$, while $K_{pt,e2,2}$ is given by

$$e^{K_{pt,e2,2}} = \frac{1}{3}\Big[(19 + 3\sqrt{33}\ )^{1/3} + 4(19 + 3\sqrt{33}\ )^{-1/3} + 1\Big] \ , \tag{5.7}$$

so that $K_{pt,e2,2} \simeq 0.60938$.

C. Effect on T_{pt} Due to Vertex Inflation

One can also deduce a general result for the effect of a uniform ℓ-fold vertex (i.e., homemorphic) inflation of the thermodynamic limit of a lattice graph $\{G\}$ with dimensionality $d \ge 2$. A generic feature of the phase transition temperature T_{pt} of a ferromagnetic spin model (above its lower critical dimensionality, so that this temperature is finite) is that, other things being equal, T_{pt} increases as a function of the vertex degree, i.e., coordination number, of the lattice. This feature is observed in approximate determinations of T_{pt} from high-temperature and low-temperature series expansions, Monte Carlo simulations, mean-field approximations, and, where available, exact solutions for T_{pt}. It is understood physically as a consequence of the fact that increasing the coordination number increases the effect of the spin-spin interactions, so that the ordering associated with the phase transition can occur in the presence of greater thermal fluctuations. On an (infinite) line, with coordination number $\kappa = 2$, a spin model with short-range interactions does not have a finite-temperature phase transition, so the lattices of interest in this section are lattices with $d \ge 2$, which necessarily have coordination number $\kappa \ge 3$ (where this minimum value, $\kappa = 3$, is realized for the honeycomb lattice). If one starts with a lattice graph $\{G\}$ with coordination number $\kappa \ge 3$, then a uniform vertex inflation,

which consists of the addition of ℓ degree-2 vertices on each edge of $\{G\}$, reduces the effective vertex degree, κ_{eff}.

Let us, for technical simplicity, consider the thermodynamic limit $\{G\}$ to be reached as the $n(G) \to \infty$ limit of a regular lattice graph G with periodic boundary conditions, and uniform vertex degree κ. Then, with $e(G) = (\kappa/2)n(G)$, the number of vertices and edges of $\mathcal{V}_\ell(G)$ are

$$n(\mathcal{V}_\ell(G)) = \left(1 + \frac{\kappa\ell}{2}\right)n(G) \tag{5.8}$$

and

$$e(\mathcal{V}_\ell(G)) = (1+\ell)e(G) \tag{5.9}$$

so that

$$\kappa_{eff}(\mathcal{V}_\ell(G)) = \left[\frac{1+\ell}{1+\frac{\kappa\ell}{2}}\right]\kappa \ . \tag{5.10}$$

The fact that the vertex inflation reduces κ_{eff} if $\kappa > 2$ is clear analytically from this result, since

$$\kappa_{eff}(\mathcal{V}_\ell(G)) < \kappa \quad \text{if } \ell \ge 1 \quad \text{and} \quad \kappa > 2 \ . \tag{5.11}$$

For example, for the square lattice, with $\kappa = 4$, one has $\kappa_{eff}(\mathcal{V}_1(sq)) = 8/3 = 2.666..$, $\kappa_{eff}(\mathcal{V}_2(sq)) = 12/5 = 2.4$, $\kappa_{eff}(\mathcal{V}_3(sq)) = 16/7 \simeq 2.286$, and so forth, with an approach to 2 from above as $\ell \to \infty$. Indeed, in general, for an arbitrary regular lattice graph G with coordination number κ,

$$\lim_{\ell\to\infty} \kappa_{eff} = 2 \tag{5.12}$$

For $\ell >> 1$, one has the Taylor series expansion

$$\kappa_{eff} = 2\left[1 + \left(1 - \frac{2}{\kappa}\right)\left\{\frac{1}{\ell} + \frac{2}{\kappa\ell^2} + O\Big(\frac{1}{\ell^3}\Big)\right\}\right] \tag{5.13}$$

Hence, for a Potts ferromagnet on a lattice graph $\{G\}$ with $d \ge 2$, a uniform ℓ-fold vertex inflation of $\{G\}$ with $\ell \ge 1$ leads to a decrease in the phase transition temperature. The same conclusion holds if one performs a vertex inflation on all edges along a subset of the lattice directions.

We illustrate the effect of vertex inflation for the q-state Potts ferromagnet on the square lattice. Let us perform a uniform vertex inflation with $\ell = 1$ on all edges of the lattice, i.e., add one degree-2 vertex to each edge of this lattice. Then, combining Eqs. (4.3) and (5.2), we find that the equation for the phase transition temperature is $v_{v,1} = \sqrt{q}$, where $v_{v,1}$ was given in Eq. (4.16). The solution for the inverse phase transition temperature $K_{pt,v,1}$ is

$$K_{pt,v,1} = \ln\left[1 + \sqrt{q}\left\{1 + \sqrt{1+\sqrt{q}}\ \right\}\right] \ . \tag{5.14}$$

Comparing this with the inverse critical temperature for the original $\{G\}$, $K_{pt} = \ln(1+\sqrt{q}\ )$, we see that $K_{pt,v,1} > K_{pt}$, in agreement with our general argument above.

The situation is more complicated with the Potts antiferromagnet; we will show that vertex inflation can either lower or raise a phase transition (critical) temperature, depending on the value of q and the lattice type. First, consider the $q = 2$ (Ising) Potts antiferromagnet on a bipartite lattice $\{G\}$. The bipartite property of $\{G\}$ means that it can be expressed as the union of an even and an odd sublattice, $\{G\} = \{G_1\} \cup \{G_2\}$, with the property that each vertex in G_1 has, as its only adjacent vertices, members of the vertex set of $\{G_2\}$ and vice versa. There is a well-known isomorphism that maps the Ising antiferromagnet on $\{G\}$ to an Ising ferromagnet on $\{G\}$, namely the simultaneous replacement $J \to -J$ and $\sigma_{v_1} \to -\sigma_{v_1}$, with σ_{v_2} unchanged, where here v_1 and v_2 denote vertices in G_1 and G_2, respectively. Thus, in this case, the effect of an ℓ-fold vertex inflation on all edges is the same for the antiferromagnetic case as for the ferromagnetic case discussed above, namely that it reduces T_{pt}. However, we next prove that vertex inflation can also have the opposite effect, of increasing a critical temperature. For this purpose, let us consider the $q = 2$ Potts (Ising) antiferromagnet on the infinite triangular lattice $\{tri\}$ (with equal negative spin-spin exchange constants in each of the three lattice directions, $J_1 = J_2 = J_3 \equiv J < 0$). This antiferromagnet is frustrated, and is only critical at $T = 0$ [27]. Now let us perform a uniform ℓ-fold vertex inflation on all edges, with $\ell = 2k+1$ odd, thereby obtaining $\{\mathcal{V}_{2k+1}(tri)\}$. This lattice, $\{\mathcal{V}_{2k+1}(tri)\}$, is bipartite, in contrast with the triangular lattice itself. Because of this, the Ising antiferromagnet is not frustrated on $\{\mathcal{V}_{2k+1}(tri)\}$, and therefore one can apply a standard Peierls-type argument to infer that it has a finite-temperature symmetry-breaking phase transition. Indeed, because $\{\mathcal{V}_{2k+1}(tri)\}$ is bipartite and because of the isomorphism mentioned above, the Ising ferromagnet and antiferromagnet can be mapped to each other, and have their respective phase transitions at the same T_{pt}. Thus, as this example shows, in contrast with the situation for the Potts ferromagnet, vertex inflation for the Potts antiferromagnet may actually raise a critical or phase transition temperature rather than lowering it, depending on q and the lattice type.

D. Invariance of Universality Class Under Edge Inflations

We recall that on two-dimensional lattices the (zero-field) q-state Potts ferromagnet with $q \le 4$ has a second-order phase transition with an associated q-dependent universality class and corresponding thermal and magnetic critical exponents that are independent of the lattice type. It is of interest to study whether vertex or edge inflation changes the universality class of this phase transition. A general result of renormalization-group analyses

of second-order phase transitions is that the universality class of a second-order phase transition (in a model that is free of complications such as competing interactions, frustration, and/or quenched disorder) depends on the lattice dimensionality, d and the symmetry group of the Hamiltonian (e.g., [28]). For the Potts model, the symmetry group of the Hamiltonian is the symmetric (permutation) group on q indices, S_q. Edge inflations of the lattice have no effect on the dimensionality or symmetry group of the Hamiltonian; hence, for the range $q \le 4$ where the two-dimensional Potts ferromagnet has a second-order phase transition, they do not change the universality class of this transition.

E. Effects of Vertex Inflation on Universality Class

We again consider the interval $q \le 4$ where the two-dimensional Potts ferromagnet has a second-order phase transition. As with edge inflation, we note that vertex inflation has no effect on the lattice dimensionality or symmetry group of the Hamiltonian, and hence, by the same argument as before, it does not change the universality class of the phase transition.

As before, the situation is more complicated for the Potts antiferromagnet, because, in contrast to the ferromagnet, its properties depend sensitively on the type of lattice. To show this, it is convenient to continue with the same illustrative example that we used above, namely the $q = 2$ Ising antiferromagnet. We will give two examples which exhibit opposite behaviors; in the first, the vertex inflation lowers the phase transition temperature and makes no change in the universality class. In the second, the vertex inflation raises the critical temperature and does change the universality class. The first example uses the Ising antiferromagnet on a bipartite lattice graph $\{G\}$ of dimensionality $d \ge 2$, where it has a phase transition at a temperature T_{pt} (with antiferromagnetic long-range order for $T < T_{pt}$). Now let us perform a uniform ℓ-fold vertex inflation on all of the edges of $\{G\}$, thereby obtaining $\{\mathcal{V}_\ell(G)\}$. By an argument similar to the one given above, this lowers T_{pt}. Since this vertex inflation does not change either the lattice dimensionality or the symmetry group $S_2 \approx \mathbb{Z}_2$, it leaves the universality class unchanged.

To show that the opposite can also happen, let us consider the (isotropic) Ising antiferromagnet on the triangular lattice. As mentioned before, because of the frustration, this model has no finite-temperature transition, but is critical at $T = 0$; the spin-spin correlation function decays asymptotically like $\langle \sigma_0 \sigma_{\vec r} \rangle \propto r^{-1/2} \cos(2\pi r/3)$ for large $r = |\vec r|$ [27]. Normally, at a second-order phase transition of a spin model on a d-dimensional lattice in which the (connected) spin-spin correlation function decays asymptotically like $\langle \sigma_0 \sigma_{\vec r} \rangle \propto r^{-(d-2+\eta)}$, one assigns the critical exponent η to this transition. Because of the oscillatory nature of the asymptotic decay of the $T = 0$ Ising antiferromagnet on the triangular lattice, this case is more complex, but the decay of the envelope curve is described by $\eta = 1/2$. Now, just as we did before, let us perform a uniform ℓ-fold vertex inflation on all edges, with $\ell = 2k + 1$ odd, thereby obtaining the bipartite lattice $\{\mathcal{V}_{2k+1}(tri)\}$. As discussed above, because $\{\mathcal{V}_{2k+1}(tri)\}$ is bipartite, the Ising antiferromagnet is not frustrated on it, and, indeed, can be mapped to the Ising ferromagnet by the mapping given in the previous subsection. Owing to this, the Ising ferromagnet and antiferromagnet on this lattice have the same phase transition temperatures, and, furthermore, the Ising antiferromagnet is automatically in the same universality class as the Ising ferromagnet, with $\eta = 1/4$ [28, 29]. Thus, in this case, the vertex inflation does change both the value of the critical temperature and the universality class of the phase transition.

VI. LONGITUDINAL HOMEOMORPHIC INFLATIONS OF FREE LADDER GRAPH

In the previous sections we have derived general relations that connect $Z(G, q, v)$ and $Z(\tilde G, q, v)$, where $\tilde G$ is obtained from G by uniform edge or vertex inflations. By Eq. (2.4), these enable one to calculate the equivalent Tutte polynomial, $T(\tilde G, x, y)$. We have also discussed the case where edge or vertex inflations are performed on all edges along a subset of lattice directions of a lattice graph. In the rest of this paper we explore the latter type of vertex inflation further. We present exact calculations of Potts/Tutte polynomials for a class of vertex (i.e., homeomorphic) inflations of a subset of the edges of ladder graphs. Our results generalize our calculations of the chromatic polynomials for these graphs with S.-H. Tsai in Refs. [9] and [7] (see also [8, 16]).

We consider the free strip of the ladder graph comprised of m squares, which we denote as S_m. Now we perform an ℓ-fold vertex inflation on all of the longitudinal edges, with $\ell = k - 2$ and $k \ge 3$, i.e., we insert $k - 2$ degree-2 vertices on each of these edges. The resultant strip graph is denoted $S_{k,m}$. The transverse edges are not affected by this operation. The original ladder graph itself is $S_{2,m}$. The numbers of vertices and edges on $S_{k,m}$ are

$$n(S_{k,m}) = 2(k-1)m + 2 \tag{6.1}$$

and

$$e(S_{k,m}) = (2k-1)m + 1 \ . \tag{6.2}$$

Using a systematic iterative application of the deletion-contraction property, we calculate the Tutte polynomial, $T(S_{k,m}, x, y)$. One way to express this is in terms of a generating function. We use a generating function

$$\Gamma(S_k, x, y; z) = \sum_{m=0}^{\infty} T(S_{k,m+1}, x, y) z^m \tag{6.3}$$

of the form

$$\Gamma(S_k, x, y; z) = \frac{a_0 + a_1 z}{1 + b_1 z + b_2 z^2} \ . \tag{6.4}$$

The denominator can be written as

$$1 + b_1 z + b_2 z^2 = (1 - \lambda_{k,0,1} z)(1 - \lambda_{k,0,2} z) \ . \tag{6.5}$$

Recall that for the circuit graph with n vertices, C_n,

$$T(C_n, x, y) = \frac{x^n + c^{(1)}}{x - 1} = y + \sum_{j=1}^{n-1} x^j \ , \tag{6.6}$$

where $c^{(1)} = q - 1 = xy - x - y$. We calculate

$$a_0 = T(C_{2k}, x, y) \ , \tag{6.7}$$

$$a_1 = -y x^{2k-1} \ , \tag{6.8}$$

$$b_0 = -[1 + T(C_{2k-1}, x, y)] \ , \tag{6.9}$$

and

$$b_1 = x^{2k-2} y \ . \tag{6.10}$$

Thus,

$$\lambda_{k,0,j} = \frac{1}{2}\Big[- b_0 \pm \sqrt{b_0^2 - 4b_1} \ \Big] \ . \tag{6.11}$$

By a generalization of Eq. (2.15) in [16] from chromatic polynomials to the full Tutte polynomial, it follows that

$$T(S_{k,m}, x, y) = \frac{(a_0 \lambda_{k,0,1} + a_1)}{(\lambda_{k,0,1} - \lambda_{k,0,2})} (\lambda_{k,0,1})^{m-1} + \frac{(a_0 \lambda_{k,0,2} + a_1)}{(\lambda_{k,0,2} - \lambda_{k,0,1})} (\lambda_{k,0,2})^{m-1} \ . \tag{6.12}$$

Note that $T(S_{k,m}, x, y)$ is symmetric under the interchange $\lambda_{k,0,1} \leftrightarrow \lambda_{k,0,2}$. It is straightforward, using Eq. (2.4), to re-express these results in terms of the Potts model partition function $Z(S_{k,m}, q, v)$; for brevity, we omit the explicit results.

VII. LONGITUDINAL HOMEOMORPHIC INFLATIONS OF CYCLIC AND MÖBIUS LADDER GRAPHS

In this section we present an exact calculation of the Tutte polynomial for longitudinal homeomorphic inflations of cyclic and Möbius ladder graphs. We start with a cyclic ladder strip of length m squares and add $k-2$ degree-2 vertices, with $k \ge 3$, to each longitudinal edge. This yields the strip graph with longitudinal homeomorphic inflation that we denote $L_{k,m}$. The corresponding Möbius strip $M_{k,m}$ is obtained by cutting the cyclic graph at any transverse edge and reattaching the ends after a vertical twist. The graphs $L_{k,m}$ and $M_{k,m}$ each have the number of vertices

$$n(L_{k,m}) = n(M_{k,m}) = 2(k-1)m \tag{7.1}$$

and the number of edges

$$e(L_{k,m}) = e(M_{k,m}) = (2k-1)m \ . \tag{7.2}$$

For a given m and for $k = 2$ these are the original cyclic and Möbius ladder strip graphs; the case $k = 3$ is the first homeomorphic inflation of these respective graphs, and so forth for higher values of k.

For the cyclic strip graph $L_{k,m}$ we calculate

$$T(L_{k,m}, x, y) = \frac{1}{x-1} \sum_{d=0}^{2} c^{(d)} \sum_{j=1}^{n_T(2,d)} (\lambda_{k,d,j})^m \tag{7.3}$$

where $c^{(0)} = 1$, $c^{(1)} = q - 1$, $c^{(2)} = q^2 - 3q + 1$, $n_T(2,0) = 2$, $n_T(2,1) = 3$, and $n_T(2,2) = 1$. The $\lambda_{k,0,j}$ for $j = 1, 2$ were given above in Eq. (6.11). For the others, we find

$$\lambda_{k,1,1} = x^{k-1} \ , \tag{7.4}$$

$$\lambda_{k,1,j} = \frac{1}{2}(u_k \pm \sqrt{r_k} \ ) \ , \quad j = 2, \ 3 \ , \tag{7.5}$$

where

$$\begin{aligned} u_k \ &= \ 2\Big(\sum_{j=0}^{k-2} x^j\Big) + x^{k-1} + y \\ &= \ \frac{x^k + x^{k-1} - 2}{x-1} + y \ , \end{aligned} \tag{7.6}$$

and

$$r_k = u_k^2 - 4x^{k-1} y \ , \tag{7.7}$$

and finally,

$$\lambda_{k,2,1} = 1 \ . \tag{7.8}$$

Although our result (7.3) (and (7.9) below) are formally rational functions in x, one easily verifies that the prefactor $1/(x-1)$ divides the expression to its right, so that

$T(L_{k,m},x,y)$ and $T(M_{k,m},x,y)$ are polynomials in x as well as y, as guaranteed by Eq. (2.2). It is again straightforward, using Eq. (2.4), to re-express these results in terms of the Potts model partition function $Z(S_{k,m},q,v)$.

For the Möbius strip $M_{k,m}$, we find that the λ's are the same as for the cyclic strip $L_{k,m}$, and the pattern of changes in the coefficients is the same as was shown in [10, 30, 31] for lattice strips without homeomorphic expansion, so that

$$T(M_{k,m},x,y) = \frac{1}{x-1}\Bigg[\sum_{j=1}^{2}(\lambda_{k,0,j})^m + c^{(1)}\Big(-(\lambda_{k,1,1})^m + \sum_{j=2}^{3}(\lambda_{k,1,j})^m\Big) - 1\Bigg] . \tag{7.9}$$

From these general expressions, one can specialize to the chromatic polynomials and the flow polynomials. Via Eq. (4.12), one readily checks that the results for the chromatic polynomials agree with those that we calculated before with S.-H. Tsai in Ref. [9]. The flow polynomials are unaffected by homeomorphic expansion, and hence coincide with those that we calculated with S.-C. Chang in Ref. [32].

VIII. VALUATIONS OF TUTTE POLYNOMIALS FOR HOMEOMORPHIC EXPANSIONS OF CYCLIC AND MÖBIUS LADDER GRAPHS

Special valuations of the Tutte polynomial yield several quantities of graph-theoretic interest. In this section we calculate these for the longitudinal homeomorphic expansions of cyclic and Möbius ladder graphs. We first recall some definitions. A tree graph is a connected graph with no circuits (cycles). A spanning tree of a graph G is a spanning subgraph of G that is also a tree. A spanning forest of a graph G is a spanning subgraph of G that may consist of more than one connected component but contains no circuits. The special valuations of interest here are (i) $T(G,1,1)=N_{ST}(G)$, the number of spanning trees (ST) of G; (ii) $T(G,2,1)=N_{SF}(G)$ the number of spanning forests (SF) of G; (iii) $T(G,1,2)=N_{CSSG}(G)$, the number of connected spanning subgraphs ($CSSG$) of G; and (iv) $T(G,2,2)=N_{SSG}(G)=2^{e(G)}$, the number of spanning subgraphs (SSG) of G. The last of these quantities is determined directly from Eq. (7.2) as

$$N_{SSG}(G_{k,m}) = 2^{(2k-1)m} , \tag{8.1}$$

where we introduce the notation $G_{k,m}$ to stand for either $L_{k,m}$ or $M_{k,m}$.

We calculate

$$\lambda_{k,0,j}\Big|_{\substack{x=2\\y=1}} = 2^{k-1}\Big[2^{k-1} \pm \Big(2^{2(k-1)}-1\Big)^{1/2}\Big] , \tag{8.2}$$

where the $\pm$ sign applies for $j=1,2$, respectively,

$$\lambda_{k,1,1}\Big|_{\substack{x=2\\y=1}} = 2^{k-1} , \tag{8.3}$$

and

$$\lambda_{k,1,j}\Big|_{\substack{x=2\\y=1}} = \frac{1}{2}\Big[3\cdot 2^{k-1} - 1 \pm \Big[(3\cdot 2^{k-1}-1)^2 - 2^{k+1}\Big]^{1/2}\Big] , \tag{8.4}$$

where the $\pm$ sign applies for $j=2,3$, respectively. Next, define the notation

$$\eta_G \equiv \eta_{G_{k,m}} = \begin{cases} +1 & \text{if } G_{k,m}=L_{k,m} \\ -1 & \text{if } G_{k,m}=M_{k,m} \end{cases} \tag{8.5}$$

In terms of these quantities, we have, for $G_{k,m}=L_{k,m}$ or $M_{k,m}$,

$$N_{SF}(G_{k,m}) = (\lambda_{k,0,1})^m + (\lambda_{k,0,2})^m + \eta_G\Big[1-(\lambda_{k,1,1})^m\Big] - \Big[(\lambda_{k,1,2})^m + (\lambda_{k,1,3})^m\Big] \tag{8.6}$$

where in Eq. (8.6) the $\lambda_{k,d,j}$'s are evaluated at $x=2$, $y=1$ as in Eqs. (8.2)-(8.4). For $k=2$, i.e. the original cyclic and Möbius strips, one checks that eq. (8.6) reduces to eq. (D.14) of our previous work [10]. As an example, for the first homeomorphic expansion, $k=3$, eq. (8.6) yields

$$N_{SF}(G_{3,m}) = [4(4+\sqrt{15}\,)]^m + [4(4-\sqrt{15}\,)]^m + \eta_G(1-2^{2m}) - \Bigg[\Bigg(\frac{11+\sqrt{105}}{2}\Bigg)^m + \Bigg(\frac{11-\sqrt{105}}{2}\Bigg)^m\Bigg] . \tag{8.7}$$

For valuations of $T(G_{k,m},x,y)$ with $x=1$ and $G_{k,m}=L_{k,m}$ or $M_{k,m}$, a useful equality is

$$\lambda_{k,0,j}\Big|_{x=1} = \lambda_{k,1,j+1}\Big|_{x=1} = \frac{1}{2}\Big[2k-1+y \pm \Big[(2k-1+y)^2 - 4y\Big]^{1/2}\Big] , \tag{8.8}$$

where the $\pm$ sign applies for $j=1,2$, respectively. For the number of spanning trees on these families on $G_{k,m}=L_{k,m}$

or $M_{k,m}$, we find

$$N_{ST}(G_{k,m}) = m(k-1)\Big[-\eta_G + \frac{1}{2}\Big\{\Big(k+\sqrt{k^2-1}\,\Big)^m + \Big(k-\sqrt{k^2-1}\,\Big)^m\Big\}\Big] \ . \tag{8.9}$$

For $k=2$, this reduces to Eq. (D.13) of [10].

Next, we introduce the shorthand notation

$$\lambda_{\pm} = \frac{1}{2}\Big(2k+1 \pm \sqrt{4k^2+4k-7}\,\Big) \tag{8.10}$$

and

$$a_{\pm} = \Big(\frac{k-1}{2}\Big)\Big[k \pm \frac{(2k^2+k-4)}{\sqrt{4k^2+4k-7}}\,\Big] \ . \tag{8.11}$$

In terms of these quantities, we find, for the number of connected spanning subgraphs,

$$N_{CSSG}(L_{k,m}) = -2 + (\lambda_+)^m + (\lambda_-)^m + \ m\Big[1-k+(\lambda_+)^{m-1}a_+ + (\lambda_-)^{m-1}a_-\Big] \tag{8.12}$$

and

$$N_{CSSG}(M_{k,m}) = N_{CSSG}(L_{k,m}) + 1 + 2(k-1)m \ . \tag{8.13}$$

For $k=2$, this reduces to Eq. (D.15) of [10].

These graphical quantities grow exponentially as a function of the strip length m and hence also n. For each quantity $N_{\mathcal{G}}$ (i.e., N_{ST}, N_{SF}, N_{CSSG}, and N_{SSG}), one can thus define a growth constant

$$w_{N_{\mathcal{G}(G)}} \equiv \lim_{m\to\infty} [N_{\mathcal{G}(G)}]^{1/n} \ . \tag{8.14}$$

In Ref. [10] we used an equivalent quantity to describe the exponential growth, viz.,

$$z_{N_{\mathcal{G}(G)}} \equiv \ln[w_{N_{\mathcal{G}(G)}}] \ . \tag{8.15}$$

We find that for a given set of subgraphs $\mathcal{G}(G)$, these growth constants are the same for the $m\to\infty$ limits of the $S_{k,m}$, $L_{k,m}$, and $M_{k,m}$ families of strip graphs, i.e.,

$$w_{N_{\mathcal{G}}(\{S_k\})} = w_{N_{\mathcal{G}}(\{L_k\})} = w_{N_{\mathcal{G}}(\{M_k\})} \ , \tag{8.16}$$

so we will just label them by $\{L_k\}$. For each type of subgraph $\mathcal{G}$, the growth constant is determined by the dominant $\lambda_{k,d,j}$, which is $\lambda_{k,0,1}$, evaluated at the respective values of (x,y). We find

$$w_{SSG(\{L_k\})} = 2^{\frac{2k-1}{2(k-1)}} \ , \tag{8.17}$$

$$w_{SF(\{L_k\})} = 2\Big[1+\Big(1-2^{-2(k-1)}\Big)^{1/2}\,\Big]^{\frac{1}{2(k-1)}} \ , \tag{8.18}$$

$$w_{CSSG(\{L_k\})} = \Big[\frac{2k+1+(4k^2+4k-7)^{1/2}}{2}\Big]^{\frac{1}{2(k-1)}} \ , \tag{8.19}$$

and

$$w_{ST(\{L_k\})} = \Big(k+\sqrt{k^2-1}\,\Big)^{\frac{1}{2(k-1)}} \ . \tag{8.20}$$

For $k=2$, i.e., the original ladder strip without homeomorphic expansion, Eqs. (8.17), (8.18), (8.19), and (8.20), together with (8.15), reduce to Eqs. (D.24), (D.22), (D.23), and (D.21) of our previous paper, Ref. [10]. Some numerical values of these growth constants obtained from the analytic results given above are displayed in Table I.

TABLE I: Values of w_{SSG}, w_{SF}, w_{CSSG}, and w_{ST} for the $m\to\infty$ limit of the $G_{k,m}$ family of graphs, where $G_{k,m}$ denotes $S_{k,m}$, $L_{k,m}$, or $M_{k,m}$.

k	w_{SSG}	w_{SF}	w_{CSSG}	w_{ST}
2	2.8284271	2.7320508	2.1357792	1.9318517
3	2.3784142	2.3689169	1.6089554	1.5537740
4	2.2449241	2.2434544	1.4360948	1.4104463
5	2.1810155	2.1807485	1.3466467	1.3318300
6	2.1435469	2.1434946	1.2908358	1.2811894
7	2.1189262	2.1189154	1.2522226	1.2454451
8	2.1015133	2.1015110	1.2236924	1.2186716
9	2.0885476	2.0885471	1.2016292	1.1977615
10	2.0785185	2.0785183	1.1839857	1.1809157
∞	2	2	1	1

IX. TUTTE POLYNOMIALS OF HAMMOCK GRAPHS

A. General Calculation

A hammock graph $H_{k,r}$ is defined as follows: start with two vertices connected by r edges (where r denotes "rope" or, more abstractly, "route"). Now add $k-2$ degree-2 vertices to each of these edges, with $k\ge 3$, so that on any rope there are a total of k vertices, including the two end-vertices. Thus, $H_{k,r}$ with $k\ge 3$ is a uniform $\ell = k-2$ fold vertex (homeomorphic) inflation of the original graph, $H_{2,r}$ with r edges connecting the two end vertices. The number of vertices, edges, and (linearly independent) cycles in this graph are

$$n(H_{k,r}) = 2 + (k-2)r \ , \tag{9.1}$$

$$e(H_{k,r}) = (k-1)r \ , \tag{9.2}$$

and

$$c(H_{k,r}) = r - 1 \ . \tag{9.3}$$

(satisfying the general relation $c(G) = e(G) + k_c(G) - n(G)$). Hence,

$$\kappa_{eff}(H_{k,r}) = \frac{2r(k-1)}{2+(k-2)r} \ . \tag{9.4}$$

An interesting feature of the $H_{k,r}$ graphs is that if $r \to \infty$ for fixed k, the combination of tne two end vertices, each with $\kappa = r$ and the $(k-2)r$ interior vertices on the "ropes", each with $\kappa = 2$, yields the result

$$\lim_{r \to \infty} \kappa_{eff}(H_{k,r}) = 2\left(\frac{k-1}{k-2}\right) \ . \tag{9.5}$$

Although $H_{k,r}$ is much simpler than the complex networks encountered in biological and social contexts, the property that for large r it has vertices of quite different degrees is also observed in complex networks [33]. The girth $g(G)$ of a graph G is the number of edges in a minimum-distance circuit in G. For $r \ge 2$, the girth of $H_{k,r}$ is

$$g(H_{k,r}) = 2k - 2 \ . \tag{9.6}$$

Before presenting our general results for $T(H_{k,r}, x, y)$, we note two special cases. First, for $r = 2$, the graph $H_{k,2}$ is just the circuit graph with $2k-2$ vertices:

$$H_{k,2} = C_{2k-2} \ . \tag{9.7}$$

Hence,

$$T(H_{k,2}, x, y) = T(C_{2k-2}, x, y) \ , \tag{9.8}$$

where $T(C_n, x, y)$ was given in Eq. (6.6). The second special case is for $k = 2$. If G is a planar graph, we denote its planar dual as G^*. From Eq. (2.2), it follows that

$$T(G, x, y) = T(G^*, y, x) \ . \tag{9.9}$$

We can apply this result here, since $H_{k,r}$ is a planar graph. Now $H_{2,r}$ is the graph consisting of two vertices connected by r edges. This is the planar dual to the circuit graph;

$$H_{2,r} = (C_r)^* \ . \tag{9.10}$$

From Eqs. (9.9), (9.10), and (6.6), it follows that

$$T(H_{2,r}, x, y) = T(C_r, y, x) = \frac{y^r + xy - x - y}{y-1} \ . \tag{9.11}$$

Proceeding to the general $H_{k,r}$ graph, we define

$$\lambda_{H,1} = \sum_{j=0}^{k-2} x^j = \frac{x^{k-1}-1}{x-1} \tag{9.12}$$

and

$$\lambda_{H,2} = T(C_{k-1}, x, y) \ . \tag{9.13}$$

Using an iterative application of the deletion-contraction relation, we calculate the Tutte polynomial of $H_{k,r}$ to be

$$T(H_{k,r}, x, y) = (\lambda_{H,1})^{r-2}\, T(C_{2k-2}, x, y) + \left[\frac{(\lambda_{H,1})^{r-2} - (\lambda_{H,2})^{r-2}}{\lambda_{H,1} - \lambda_{H,2}}\right](\lambda_{H,2})^2 \ . \tag{9.14}$$

B. Chromatic and Flow Polynomials

Using Eq. (4.12), one readily verifies that for the case $x = 1-q$, $y = 0$, the Tutte polynomial (9.14) yields the chromatic polynomial for $H_{k,r}$ that we calculated in (Eq. (3.7) of) Ref. [7].

One can also discuss other special cases. For the flow polynomial $F(H_{k,r}, q)$, we set $x = 0$, $y = 1-q$ and, with eq. (4.19), we have

$$F(H_{k,r}, q) = q^{-1}[(q-1)^r + (q-1)(-1)^r] \ . \tag{9.15}$$

This is independent of k, in accordance with the general result (4.20). If G is a planar graph and G^* is its planar dual, then the flow and chromatic polynomials satisfy the relation

$$F(G, q) = q^{-1} P(G^*, q) \ . \tag{9.16}$$

We observe that $F(H_{k,r}, q) = q^{-1} P(C_r, q)$, in accord with (4.20) and the fact that $H_{2,r} = (C_r)^*$ (cf. Eq. (9.10)).

C. Reliability Polynomial

A communication network, such as the internet, can be represented by a graph, with the vertices of the graph

representing the nodes of the network and the edges of the graph representing the communication links between these nodes. In realistic networks, both the nodes and the links between them are imperfect, and fail to operate. One common measure of the reliability of the network is the probability that there is a working communications route between any node and any other node. This is the all-terminal reliability function. This is commonly modeled by a simplification in which one assumes that each node and link are operating with respective probabilities p_{node} and p_{link}. As probabilities, p_{node} and p_{link} lie in the interval [0,1]. The dependence of the all-terminal reliability function $R_{tot}(G, p_{node}, p_{link})$ on p_{node} is an overall factor of $(p_{node})^n$; i.e., $R_{tot}(G, p_{node}, p_{link}) = (p_{node})^n R(G, p_{link})$. Thus, the difficult part of the calculation of $R_{tot}(G, p_{node}, p_{link})$ is the determination of $R(G, p_{link})$. For notational brevity, we set $p_{link} \equiv p$. The function $R(G, p)$ is given by

$$R(G,p) = \sum_{\tilde{G} \subseteq G} p^{e(\tilde{G})} (1-p)^{e(G)-e(\tilde{G})} , \tag{9.17}$$

where $\tilde{G}$ is a connected spanning subgraph of G. Clearly, $R(G,p)$ is a monotonically increasing function of $p \in [0,1]$ with the boundary values $R(G,0) = 0$ and $R(G,1) = 1$. $R(G,p)$ can be related to a special case of the Tutte polynomial, evaluated with $x = 1$ (guaranteeing that $\tilde{G}$ is a connected spanning subgraph of G) and $y = 1/(1-p)$. This relation is

$$R(G,p) = p^{n-1}(1-p)^{e(G)+1-n}\, T(G, 1, \frac{1}{1-p}) . \tag{9.18}$$

As in our previous calculations for lattice strips [34], we can thus obtain reliability polynomials as special cases of Tutte polynomials. Using our calculation in Eq. (9.14) of $T(H_{k,r}, x, y)$ with $x = 1$ and $y = 1/(1-p)$ in Eq. (9.18), we have calculated $R(H_{k,r}, p)$. For $r = 2$, we have

$$R(H_{k,2}, p) = R(C_{2k-2}, p) = p^{2k-3}[p + 2(k-1)(1-p)] , \tag{9.19}$$

in accord with Eq. (9.7). For $k = 2$, i.e., the case of no homeomorphic expansion, we find, in accord with Eq. (9.10), that

$$R(H_{2,r}, p) = R((C_r)^*, p) = 1 - (1-p)^r . \tag{9.20}$$

In general, we find that for fixed $p \in (0,1)$ and fixed r, $R(H_{k,r}, p)$ is a monotonically decreasing function of k. This can be interpreted as a consequence of the fact that as k increases, the girth $g(H_{k,r})$ increases (cf. Eq. (9.6)), and hence there is a greater likelihood that one of the communication links along the minimum-distance path and other paths between two nodes is not operating.

In contrast, for fixed $p \in (0,1)$ and fixed $k \ge 3$, we find a variety of behaviors for $R(H_{k,r}, p)$ as a function of r. To illustrate this, we take $k = 3$ and compare a few pairs of values $(r, r') = (r, 2)$. We calculate

$$R(H_{3,2}, p) = p^3(4 - 3p) , \tag{9.21}$$

$$R(H_{3,3}, p) = p^4(7p^2 - 18p + 12) , \tag{9.22}$$

$$R(H_{3,4}, p) = p^5(4 - 3p)(5p^2 - 12p + 8) . \tag{9.23}$$

and

$$R(H_{3,5}, p) = p^6(31p^4 - 150p^3 + 280p^2 - 240p + 80) \tag{9.24}$$

Hence, for the pairs $(r, r') = (r, 2)$,

$$R(H_{3,3}, p) - R(H_{3,2}, p) = p^3(1-p)^2(7p - 4) , \tag{9.25}$$

$$R(H_{3,4}, p) - R(H_{3,2}, p) = p^3(1-p)^2(4-3p)(5p^2 - 2p - 1) , \tag{9.26}$$

and

$$R(H_{3,5}, p) - R(H_{3,2}, p) = p^3(1-p)^2(31p^5 - 88p^4 + 73p^3 - 6p^2 - 5p - 4) \tag{9.27}$$

As is evident from the difference in Eq. (9.25),

$$R(H_{3,3}, p) > R(H_{3,2}, p) \quad \text{if} \quad 1 > p > \frac{4}{7} \simeq 0.571 , \tag{9.28}$$

while

$$R(H_{3,3}, p) < R(H_{3,2}, p) \quad \text{if} \quad 0 < p < \frac{4}{7} . \tag{9.29}$$

Similarly, Eq. (9.26) shows that

$$R(H_{3,4}, p) > R(H_{3,2}, p) \quad \text{if} \quad 1 > p > \frac{1+\sqrt{6}}{5} \simeq 0.690 , \tag{9.30}$$

while

$$R(H_{3,4}, p) < R(H_{3,2}, p) \quad \text{if} \quad 0 < p < \frac{1+\sqrt{6}}{5} . \tag{9.31}$$

Similarly, $R(H_{3,5}, p) > R(H_{3,2}, p)$ for $1 > p > 0.8418$, and $R(H_{3,5}, p) < R(H_{3,2}, p)$ for $0.8418 > p > 0$ (where the crossover value of p is a root of the quintic in Eq. (9.27) quoted to four significant figures). Note how, for a fixed value of k, the value of p beyond which $R(H_{3,r}, p)$ is greater than $R(H_{3,2}, p)$ increases with r, from 0.571 for $r = 3$ to 0.690 for $r = 4$ to 0.842 for $r = 5$ (to three figures accuracy), and so forth for higher values of r. We observe similar behavior for $R(H_{k,r}, p) - R(H_{k,2}, p)$ as a function of r for higher values of k.

However, we also have

$$R(H_{3,4}, p) - R(H_{3,3}, p) = -p^4(1-p)^2(15p^2 - 26p + 12) , \tag{9.32}$$

and

$$R(H_{3,5}, p) - R(H_{3,4}, p) = -p^5(1-p)^2(-31p^3 + 88p^2 - 88p + 32) . \tag{9.33}$$

These comparisons provide a contrasting type of behavior, since

$$R(H_{3,5}, p) < R(H_{3,4}, p) < R(H_{3,3}, p) \quad \forall\, p \in (0,1) . \tag{9.34}$$

and so forth with $R(H_{3,r}, p)$ for larger values of r. Thus, for these cases, for all $p \in (0,1)$, $R(H_{3,r}, p)$ is a monotonically decreasing function of r as r increases above 3.

D. Some Graphical Quantities

We next discuss special valuations of $T(H_{k,r},x,y)$ that yield quantities of graph-theoretic interest. For $x=1$, the following result is convenient:

$$T(H_{k,r},1,y) = (k-1)^{r-2}(2k+y-3) + \left[\frac{(k+y-2)^{r-2}-(k-1)^{r-2}}{y-1}\right](k+y-2)^2 \ . \tag{9.35}$$

In addition to

$$N_{SSG}(H_{k,r}) = 2^{(k-1)r} \ , \tag{9.36}$$

we compute

$$N_{ST}(H_{k,r}) = r(k-1)^{r-1} \ , \tag{9.37}$$

$$N_{SF}(H_{k,r}) = (2^{k-1}-1)^{r-2}\Big[2^{2(k-1)}-1+(r-2)(2^{k-1}-1)\Big] \ , \tag{9.38}$$

and

$$N_{CSSG}(H_{k,r}) = k^r - (k-1)^r \ . \tag{9.39}$$

Since these are two-parameter families of graphs, one can consider the limits (i) $r \to \infty$ with fixed finite k, and (ii) $k \to \infty$ with fixed finite r. We calculate the growth constants for each of these. The cases of interest here are those with $k \ge 3$, i.e., those with homeomorphic expansion. For these we find that for the limit (i) ,

$$(i): \ w_{SSG} = 2^{\frac{k-1}{k-2}} \ , \tag{9.40}$$

$$(i): \ w_{ST} = (k-1)^{\frac{1}{k-2}} \ , \tag{9.41}$$

$$(i): \ w_{SF} = (2^{k-1}-1)^{\frac{1}{k-2}} \ , \tag{9.42}$$

and

$$(i): \ w_{CSSG} = k^{\frac{1}{k-2}} \ . \tag{9.43}$$

For the limit (ii), we calculate

$$(ii): \ w_{SSG} = w_{SF} = 2 \tag{9.44}$$

and

$$(ii): \ w_{ST} = w_{CSSG} = 1 \ . \tag{9.45}$$

X. CONCLUSIONS

In conclusion, in this paper have have derived exact relations between the Potts model partition function, or equivalently, the Tutte polynomial, for a graph G and for a graph $\tilde{G}$ obtained from G by edge or vertex inflation. An analysis was given of some physical effects of uniform ℓ-fold edge and vertex inflations. We have presented exact calculations of the Tutte polynomials for free, cyclic, and Möbius ladder families of graphs $S_{k,m}$, $L_{k,m}$, and $M_{k,m}$ of length m, with $\ell = k-2$ vertex inflation on all longitudinal edges. We have given similar results for the Tutte polynomial of the family $H_{k,r}$ of hammock graphs. Our present results generalize our previous calculations in Refs. [7, 9, 10]. As one application, we have calculated reliability polynomials for the hammock graphs and analyzed their properties as a function of k and r. In addition, we have used our calculations to compute the number of spanning trees, spanning forests, and connected spanning subgraphs on these families and to determine their asymptotic behavior as the number of vertices goes to infinity.

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